

ON THE NUMBER OF EDGE-COLOURINGS OF REGULAR BIPARTITE GRAPHS

A. SCHRIJVER*

Instituut voor Actuarie en Econometrie, Universiteit van Amsterdam, Jodenbreestraat 23, Amsterdam, The Netherlands

Received 21 January 1980

Revised 13 April 1981

Let $f(k)$ be the largest number such that each k -regular bipartite graph with $2n$ vertices has at least $f(k)^n$ 1-factorizations. We prove that $f(k) \leq k!^2/k^k$, and that equality holds if k contains no other prime factors than 2 and 3. We conjecture equality for each k .

Introduction

We are interested in lower bounds for the number of edge-colourings (1-factorizations) of regular bipartite graphs. Let $f(k, n)$ be the smallest possible number of k -edge-colourings of a k -regular bipartite graph with $2n$ vertices. (A k -edge-colouring is an ordered partition of the edge set into perfect matchings. In this paper, graphs are allowed to have multiple edges.) In particular, we are interested in the largest possible number $f(k)$ such that $f(k, n) \geq f(k)^n$ for each n , i.e., in

$$f(k) := \inf_{n \in \mathbb{N}} \sqrt[n]{f(k, n)} = \lim_{n \rightarrow \infty} \sqrt[n]{f(k, n)} \quad (1)$$

(the second equality follows from "Fekete's lemma", as $f(k, n_1 + n_2) \leq f(k, n_1) \cdot f(k, n_2)$).

We show that

$$f(k) \leq k!^2/k^k, \quad (2)$$

and that equality holds if $k = 2^a 3^b$. We conjecture that equality holds for each natural number k . This would follow from a conjecture made in [3]: the number of perfect matchings in a k -regular bipartite graph with $2n$ vertices is at least

$$\left(\frac{(k-1)^{k-1}}{k^{k-2}} \right)^n. \quad (3)$$

As corollaries we derive some results on the number of latin squares and on permanents, namely Bang's [1] lower bound of e^{-n} for permanents of doubly

* Research supported by the Netherlands Organisation for the Advancement of Pure Research (Z.W.O.).

stochastic matrices of order n . In fact the methods in this paper have been inspired to a large extent by the paper of Bang and by discussions with W.G. Valiant on a related paper of Friedland [2].

Results

Theorem 1. $f(k, n) \leq k!^{2n} n!^k / (nk)!$.

Proof. Let $X = \{1, \dots, kn\}$, and let Π be the collection of all ordered partitions $\mathcal{A} = (A_1, \dots, A_n)$ of X into n classes of size k . If \mathcal{A} and \mathcal{B} are in Π , denote by $c(\mathcal{A}, \mathcal{B})$ the number of partitions $\mathcal{C} = (C_1, \dots, C_k)$ of X into k classes of size n such that

$$|A_i \cap C_j| = |B_i \cap C_j| = 1, \quad (4)$$

for $i = 1, \dots, n$; $j = 1, \dots, k$ (i.e., each C_j is a common system of distinct representatives (SDR) for \mathcal{A} and \mathcal{B}). It is easy to see that $c(\mathcal{A}, \mathcal{B})$ is the number of k -edge-colourings of the k -regular bipartite graph with vertices, say, $a_1, \dots, a_n, b_1, \dots, b_n$, where a_i and b_j are connected by $|A_i \cap B_j|$ edges, for $i, j = 1, \dots, n$. In particular, $c(\mathcal{A}, \mathcal{B}) \geq f(k, n)$.

Now fix some \mathcal{A} in Π , and consider the sum

$$\sum_{\mathcal{B} \in \Pi} c(\mathcal{A}, \mathcal{B}). \quad (5)$$

This sum may be evaluated in two ways. First

$$\sum_{\mathcal{B} \in \Pi} c(\mathcal{A}, \mathcal{B}) \geq |\Pi| \cdot f(k, n) = \frac{(nk)!}{k!^n} \cdot f(k, n). \quad (6)$$

Alternatively, there are $k!^n$ possible partitions $\mathcal{C} = (C_1, \dots, C_k)$ of X with $|A_i \cap C_j| = 1$ for $i = 1, \dots, n$; $j = 1, \dots, k$. For each such partition there are $n!^k$ partitions \mathcal{B} in Π such that $|B_i \cap C_j| = 1$, for $i = 1, \dots, n$; $j = 1, \dots, k$. So the sum (5) is equal to $k!^n \cdot n!^k$. Combining this with (6) yields the required upper bound for $f(k, n)$ \square

Corollary 1a. $f(k) \leq k!^2 / k^k$.

Proof. Apply Stirling's asymptotical formula for $n \rightarrow \infty$ to Theorem 1. \square

Theorem 2

$$f(kl, n) \geq \left(\frac{(kl)!}{k!^l \cdot l!^k} \right)^{2n} \cdot f(l, kn) \cdot f(k, n)^l.$$

Proof. Let $G = (V, E)$ be a kl -regular bipartite graph with $2n$ vertices, having

exactly $f(kl, n)$ kl -edge-colourings. Consider all possible graphs G' arising from G as follows. Each vertex of G is replaced by k new vertices, while each edge e of G is replaced by one new edge connecting two of the new vertices replacing the endpoints of the original edge e , in such a way that the new graph g' is l -regular. So the number of graphs G' arising from G in this way is equal to

$$\left(\frac{(kl)!}{l!^k}\right)^{2n}, \tag{7}$$

since for each vertex v of G we have to partition, arbitrarily, the edges incident with v into k classes of size l .

Let Π be the collection of all partitions (E_1, \dots, E_l) of the edge set of G into l classes such that each class E_i induces a k -regular subgraph of G . Now any l -edge-colouring (E_1, \dots, E_l) of a derived graph G' yields a partition in Π . Conversely, each partition in Π arises in this way from an l -edge-colouring of $k!^{2ln}$ graphs G' . Hence, by (7),

$$k!^{2ln} \cdot |\Pi| \geq \left(\frac{(kl)!}{l!^k}\right)^{2n} \cdot f(l, kn). \tag{8}$$

Now each class E_i of a partition \mathcal{E} in Π can be refined to a k -edge-colouring of the graph (V, E_i) in at least $f(k, n)$ ways. So \mathcal{E} can be refined in at least $f(k, n)^l$ ways to a kl -edge-colouring of G . Therefore, the total number $f(kl, n)$ of kl -edge-colourings of G is at least $|\Pi| \cdot f(k, n)^l$, that is, by (8), at least the required lower bound. \square

Corollary 2a

$$f(kl) \geq \left(\frac{(kl)!}{k!^l \cdot l!^k}\right)^2 \cdot f(k)^l \cdot f(l)^k.$$

Proof. Directly from the definition of $f(k)$ and Theorem 2. \square

A natural function seems to be

$$g(k) := \left(\frac{k!^2}{f(k)}\right)^{1/k}. \tag{9}$$

Corollary 1a states that $g(k) \geq k$, while Corollary 2a asserts that $g(kl) \leq g(k) \cdot g(l)$.

Corollary 2b. If $f(k) = k!^2/k^k$ for $k = k_1$ and for $k = k_2$, then also for $k = k_1k_2$.

Proof. Directly from Corollaries 1a and 2a. \square

Corollary 2c. If $k = 2^a 3^b$, then $f(k) = k!^2/k^k$.

Proof. By Corollaries 1a and 1b it suffices to show that $f(2) \geq 1$ and $f(3) \geq \frac{4}{3}$. The former inequality is trivial, while the latter follows from the result of Voorhoeve [4] that the permanent of a nonnegative, integral matrix of order n with line sums 3 is at least $(\frac{4}{3})^n$, i.e., that the number of perfect matchings in a 3-regular bipartite graph with $2n$ vertices is at least $(\frac{4}{3})^n$. Hence $f(3, n) \geq (\frac{4}{3})^n$. \square

Let $p(k, n)$ be the smallest possible permanent of a nonnegative, integral matrix of order n with line sums k , i.e., the smallest possible number of perfect matchings of a k -regular bipartite graph with $2n$ vertices. Van der Waerden's conjecture (recently proved by Falikman (added in proof)) says

$$p(k, n) \geq \left(\frac{k}{n}\right)^n \cdot n!, \quad (10)$$

while in [3] it was conjectured that

$$p(k, n) \geq \left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n. \quad (11)$$

(Note that the first bound is asymptotical for $k \rightarrow \infty$, while the second one is asymptotical for $n \rightarrow \infty$.) Now one easily sees that

$$f(k, n) \geq p(k, n) \cdot p(k-1, n) \cdot \cdots \cdot p(1, n). \quad (12)$$

So conjecture (11) implies our conjecture that

$$f(k, n) \geq \left(\frac{k!^2}{k^k}\right)^n. \quad (13)$$

Let $L(n)$ denote the number of latin squares of order n . Since $L(n)$ is equal to the number of n -edge-colourings of the complete bipartite graph $K_{n,n}$, we know that $L(n) \geq f(n, n) \geq p(n, n) \cdot p(n-1, n) \cdot \cdots \cdot p(1, n)$. Both Van der Waerden's conjecture (10) and conjecture (11) imply that

$$L(n) \geq n!^{2n}/n^{n^2} \quad (14)$$

(cf. Wilson [5]), so both conjectured lower bounds support some evidence to each other. In fact, the lower bound (14) indeed can be proved if n has no other prime factors than 2 and 3.

Corollary 2d. *If $n = 2^a 3^b$, then there are at least $n!^{2n}/n^{n^2}$ latin squares of order n .*

Proof. Directly from Corollary 2c. \square

Finally we derive the following result of Bang [1].

Corollary 2e. *The permanent of a doubly stochastic matrix of order n is at least e^{-n} .*

Proof. Since the dyadic doubly stochastic matrices form a dense subset of the space of all doubly stochastic matrices it suffices to prove the lower bound for dyadic matrices only. So let $M = (m_{ij})$ be a dyadic doubly stochastic matrix of order n . Let u be a natural number such that $2^u M$ is integral, and let for each $t \geq u$, G_t be the 2^t -regular bipartite graph with vertices, say, $a_1, \dots, a_n, b_1, \dots, b_n$, where there are $2^t m_{ij}$ edges connecting a_i and b_j , for $i, j = 1, \dots, n$. This means that for $t \geq u$, G_t arises from G_u by replacing each edge by 2^{t-u} parallel edges. Hence the number μ of perfect matchings in G_u is equal to

$$\mu = 2^{un} \cdot \text{per } M. \quad (15)$$

Moreover, the number γ_t of 2^t -edge-colourings of G_t satisfies

$$\gamma_t \leq \mu^{2^t} \cdot (2^{t-u})!^{2^{un}}, \quad (16)$$

since each colouring is determined by 2^t perfect matchings in G_u , together with an ordering of the 2^{t-u} "copies" in G_t of each of the edges of G_u . But by Corollary 2c γ_t also satisfies

$$\gamma_t \geq (2^t)!^{2^n} / 2^{t^2 n}. \quad (17)$$

Combining (15), (16) and (17) we obtain a lower bound for $\text{per } M$, which tends to e^{-n} if $t \rightarrow \infty$ by Stirling's asymptotical formula. \square

Corollary 2f. *If G is a k -regular bipartite graph with $2n$ vertices, then G has at least $(k/e)^n$ perfect matchings, i.e., $p(k, n) \geq (k/e)^n$.*

Proof. Directly from Corollary 2e. \square

Corollary 2g. $f(k) \geq k!/e^k$.

Proof. Directly from (12) and Corollary 2f. \square

Acknowledgement

I thank Prof. L. Lovász for stimulating discussions.

References

- [1] T. Bang, On matrix-functions giving a good approximation to the Van der Waerden permanent conjecture, Preprint 30, Matematisk Institut, København, 1979.
- [2] S. Friedland, A lower bound for the permanent of a doubly stochastic matrix, *Annals of Math.* 110 (1979) 167-176.
- [3] A. Schrijver and W.G. Valiant, On lower bounds for permanents, *Indag. Math.* 42 (1980) 425-427.
- [4] M. Voorhoeve, A lower bound for the permanent of certain $(0, 1)$ -matrices, *Indag. Math.* 41 (1979) 83-86.
- [5] R.M. Wilson, Nonisomorphic Steiner Triple Systems, *Math. Z.* 135 (1974) 303-313.